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Some anisotropic non-static perfect fluid cosmological models in general relativity

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Abstract. In this paper we have derived perfect fluid cosmological models which are anisotropic, non-static and have homogeneous distributions of density and pressure. Various physical properties of the models have been explored.

1. Introduction

Non-static universes play an important role in the understanding of phenomena of cosmological and astrophysical significance. Homogeneous and isotropic cosmological models have been extensively studied by several authors. Non-static anisotropic universes have attracted the attention of many authors and both homogeneous and inhomogeneous models have been constructed. Heckmann and Schucking (1962) have derived general models representing incoherent matter. Further work in this line has been done by Singh and Singh (1968), Jacobs (1968), Ellis and MacCallum (1969), Singh and Abdussattar (1973) and Roy and Singh (1976) to name but a few.

In this paper we have considered perfect fluid cosmological models which are non-static and have cylindrical symmetry. Additional assumptions regarding the behaviour of the transverse scale factors lead to homogeneous universes. Some exact solutions have been obtained and their physical features have been examined.

We consider the metric in the form

$$ds^2 = -dt^2 + A^2\,dx^2 + B^2\,dy^2 + C^2\,dz^2 \tag{1}$$

where A, B and C are functions of x and t alone. The field equations

$$-8\pi G T_i^j = R_i^j - \tfrac{1}{2}R\delta_i^j + \Lambda\delta_i^j \tag{2}$$

with

$$T_i^j = (\varepsilon + p)v_i v^j + p\delta_i^j \tag{3}$$

for the metric (1) reduce to

$$-\frac{B_{44}}{B} - \frac{C_{44}}{C} - \frac{B_4C_4}{BC} + \frac{1}{A^2}\frac{B_1C_1}{BC} + \Lambda = 8\pi G p \tag{4}$$

$$-\frac{A_{44}}{A} - \frac{C_{44}}{C} - \frac{A_4C_4}{AC} + \frac{1}{A^2}\left(\frac{C_{11}}{C} - \frac{A_1C_1}{AC}\right) + \Lambda = 8\pi G p \tag{5}$$

$$-\frac{A_{44}}{A} - \frac{B_{44}}{B} - \frac{A_4B_4}{AB} + \frac{1}{A^2}\left(\frac{B_{11}}{B} - \frac{A_1B_1}{AB}\right) + \Lambda = 8\pi G p \tag{6}$$

$$\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{A_4C_4}{AC}-\frac{1}{A^2}\left(\frac{B_{11}}{B}+\frac{C_{11}}{C}+\frac{B_1C_1}{BC}-\frac{A_1B_1}{AB}-\frac{A_1C_1}{AC}\right)-\Lambda=8\pi G\varepsilon \tag{7}$$

$$-\frac{B_{14}}{B}-\frac{C_{14}}{C}+\frac{A_4}{A}\left(\frac{B_1}{B}+\frac{C_1}{C}\right)=0 \tag{8}$$

the coordinates being assumed to be comoving. The suffixes 1 and 4 after the symbols A, B and C indicate differentiations with respect to x and t respectively. Equations (4)–(8) lead to

$$\frac{2\mu_{14}}{\mu}-\frac{\mu_1\mu_4}{\mu^2}+\frac{\nu_1\nu_4}{\nu^2}=2\frac{A_4}{A}\frac{\mu_1}{\mu} \tag{9}$$

$$\left(\frac{\nu_4}{\nu}\right)_4+\left(\frac{A_4}{A}+\frac{\mu_4}{\mu}\right)\frac{\nu_4}{\nu}=\frac{1}{A^2}\left[\left(\frac{\nu_1}{\nu}\right)_1-\left(\frac{A_1}{A}-\frac{\mu_1}{\mu}\right)\frac{\nu_1}{\nu}\right] \tag{10}$$

$$2\frac{A_{44}}{A}-\left(\frac{\nu_4}{\nu}\right)_4-\frac{\mu_{44}}{\mu}+\frac{A_4}{A}\frac{\mu_4}{\mu}-\left(\frac{A_4}{A}+\frac{\mu_4}{\mu}\right)\frac{\nu_4}{\nu}$$
$$=\frac{1}{A^2}\left[\left(\frac{\mu_1}{\mu}\right)_1-\left(\frac{\nu_1}{\nu}\right)_1-\left(\frac{A_1}{A}+\frac{\nu_1}{\nu}\right)\left(\frac{\mu_1}{\mu}-\frac{\nu_1}{\nu}\right)\right] \tag{11}$$

where $\mu=BC$ and $\nu=B/C$. We shall consider two cases: I, $\nu_1=0$, and II, $\mu_1=0$, $\nu_1\neq 0$.

2. Case I: $\nu_1=0$

In case I there are two possibilities, namely, $\mu_1=0$ and $\mu_1\neq 0$. In the first case equation (9) is identically satisfied. Equation (10) then shows that A is of the form of the product of a function of x and a function of t. By an obvious transformation the metric reduces to that of Bianchi type I. This metric has been widely discussed in cosmological context. In the second case when $\mu_1\neq 0$ from equation (9) we obtain

$$A\mu(\nu_4/\nu)=F(x). \tag{12}$$

From equation (10) we also obtain

$$\mu^{-1/2}(\mu_1/A)=Q(x). \tag{13}$$

From equations (12) and (13) we obtain

$$\mu=(\alpha(x)(\nu/\nu_4)+\gamma(t))^{2/3} \tag{14}$$

where $\alpha_1=\frac{3}{2}FQ$. Substitution in (11) leads to

$$A=\frac{1}{K}\left(\frac{\nu}{\nu_4}\right)^{1/3}\frac{\alpha_1}{\alpha}\qquad \mu=\alpha^{2/3}\left(\frac{\nu}{\nu_4}\right)^{2/3}$$

where K is a constant. The metric (1) therefore reduces to the form

$$ds^2=-R^6\,dT^2+R^2\,dx^2+R^2\,e^{-2x}(e^{mT}\,dy^2+e^{-mT}\,dz^2) \tag{15}$$

by a suitable transformation of coordinates. The metric (15) corresponds to the orthogonal Bianchi type V. Bianchi type V universes are a generalisation of FRW universes with constant negative curvature since the three surfaces $T=$ constant are of constant negative curvature. These belong to class B with $\eta^{\alpha}_{\alpha} = 0$ in the notation of Ellis and MacCallum (1969). Bianchi type V universes have been extensively studied by various authors. Grishchuk *et al* (1969) have considered a perfect fluid with $V^1 \neq 0$. The case with $V^1 \neq 0$, $V^3 \neq 0$ was considered by Ruzmaikina and Ruzmaikin (1969). Matzner (1969) considered a model which includes viscosity. Shikin (1975) obtained solutions with dust and radiation. Models with multifluid components were studied by Hughston and Shepley (1970). Questions relating to asymptotic behaviour and singularities have been discussed by MacCallum (1971) and Collins and Ellis (1979). Collins has also studied Bianchi type V models as plane autonomous systems (Collins 1971).

The equation $T^{ij}_{;j}u^i = 0$ in the case of a perfect fluid leads for the metric (15) to

$$3(\varepsilon + p)\dot{R}/R + \dot{\varepsilon} = 0 \tag{16}$$

where a dot denotes differentiation with respect to $\tau = \int R^3 \, dT$. If the equation of state is $p = p(\varepsilon)$ then we get on integration

$$R^3 = \beta \, e^{f(\varepsilon)} \tag{17}$$

where β is a constant and

$$f(\varepsilon) = -\int \frac{d\varepsilon}{\varepsilon + p}.$$

For the metric (15) we have

$$\dot{R}^2/R^2 = \tfrac{1}{3}(8\pi G\varepsilon + m^2/4R^6 + 3/R^2 + \Lambda) \tag{18}$$

so that

$$\tau = \frac{1}{3}\int \frac{df}{d\varepsilon}\left(\frac{8\pi G}{3}\varepsilon + \frac{m^2}{12\beta^2} e^{-2f} + \beta^{-2/3} e^{-2f/3} + \frac{\Lambda}{3}\right) d\varepsilon + \tau_0 \tag{19}$$

where τ_0 is a constant of integration. Equations (17) and (19) give the parametric form for the model. In the case of the barotropic equation of state $p = (\gamma - 1)\varepsilon$ where $1 \leqslant \gamma = \text{constant} \leqslant 2$ we have

$$\varepsilon = \varepsilon_0 R_0^{3\gamma}/R^{3\gamma} \tag{20}$$

where ε_0 and R_0 are the present values of density and radius of the universe. Equation (18) then leads to

$$\frac{\dot{R}^2}{R^2} = \frac{m^2}{12R^6} + \frac{1}{R^2} + \frac{\Lambda}{3} + \frac{8\pi G}{3}\frac{\varepsilon_0 R_0^{3\gamma}}{R^{3\gamma}}. \tag{21}$$

For $\Lambda < 0$, $\dot{R}$ vanishes for only one real value of R and $\ddot{R} < 0$ at that instant. Hence in this case the model expands from an initial singularity, and after reaching a maximum it recontracts to a second singularity (cf MacCallum 1971). When $\Lambda \geqslant 0$, $\dot{R} > 0$ so that it is an ever expanding model. A simple calculation tells us that when $\Lambda > 0$ the rate of expansion $\dot{R}$ which is infinite at the beginning attains a minimum at $R = R_1$ given by

$$\frac{m^2}{R_1^6} = 2\Lambda - \frac{8\pi G\varepsilon_0 R_0^{3\gamma}(3\gamma - 2)}{R_1^{3\gamma}} \tag{22}$$

and thereafter it is speeded up. MacCallum has given a qualitative discussion of the horizon structure of the homogeneous models. For the present model we can verify the existence of both particle and event horizons. To see this consider a ray of light emitted parallel to the X axis from the point whose coordinates are (X_1, Y_1, Z_1) by a comoving particle at time τ_1 and received by an observer whose x coordinate is zero at time τ. Then

$$X_1 = \int_{\tau_1}^{\tau} \frac{d\tau}{R}$$

$$= \int_{R(\tau_1)}^{R(\tau)} \frac{R\,dR}{(\frac{1}{12}m^2 + \frac{8}{3}\pi G\varepsilon_0 R_0^{3\gamma} R^{6-3\gamma} + R^4 + \frac{1}{3}\Lambda R^6)^{1/2}}. \tag{23}$$

The integral converges as $\tau_1 \to 0$, i.e. $R(\tau_1) \to 0$. This shows that a particle horizon exists and its proper distance is given by

$$d_H(\tau) = R(\tau) \int_0^R \frac{R\,dR}{(\frac{1}{12}m^2 + \frac{8}{3}\pi G\varepsilon_0 R_0^{3\gamma} R^{6-3\gamma} + R^4 + \frac{1}{3}\Lambda R^6)^{1/2}}. \tag{24}$$

Clearly $d_H(\tau)$ increases less or more rapidly than $R(\tau)$ according to whether $\Lambda \gtrless 0$. A similar analysis shows the existence of particle horizons in the Y and Z directions. The integral in (23) converges as $R(\tau) \to \infty$ when $\Lambda > 0$, so that in this case the universe has an event horizon.

In the case of a dust filled model for which $\gamma = 1$, the deceleration parameter q is given by

$$q = \frac{\frac{1}{6}m^2 + \frac{4}{3}\pi G R_0^3 R^3 \varepsilon_0 - \frac{1}{3}\Lambda R^6}{(\frac{1}{12}m^2 + \frac{8}{3}\pi G\varepsilon_0 R_0^3 R^3 + R^4 + \frac{1}{3}\Lambda R^6)}. \tag{25}$$

Hence for

$$\Lambda > 0 \qquad q \to -1 \qquad \text{as} \quad R \to \infty$$

$$\Lambda = 0 \qquad q \to 0 \qquad \text{as} \quad R \to \infty$$

$$\Lambda < 0 \qquad q \to \infty \qquad \text{as} \quad R \to R'$$

where R' is the maximum radius of the universe. In the third case we have

$$q < 3 \qquad \text{for } 0 < R < R''$$

where R'' is given by

$$\tfrac{1}{12}m^2 + \tfrac{20}{3}\pi G\varepsilon_0 R_0^3 R''^3 + 3R''^4 + \tfrac{4}{3}\Lambda R''^6 = 0. \tag{26}$$

Since q_0, the present value of the deceleration parameter, is nearly equal to 1 we conclude that either our universe is infinitely expanding or if it is expanding to a finite dimension then the present epoch is sufficiently earlier than the time when $R = R'$. Since σ/Θ and $8\pi G\varepsilon/\Theta^2$ tend to zero as $R \to \infty$ for $\Lambda \geq 0$, the corresponding model asymptotically tends to become isotropic (Szafron 1977). The constant m and the present linear dimension R_0 can be expressed in terms of the present Hubble parameter H_0, deceleration parameter q_0 and density ε_0. These are given by

$$R_0^2 = \frac{2}{(2 - q_0)H_0^2 - 4\pi G\varepsilon_0 - \Lambda} \tag{27}$$

$$m^2 = \frac{16(3q_0 H_0^2 + \Lambda - 4\pi G\varepsilon_0)}{[(2 - q_0)H_0^2 - 4\pi G\varepsilon_0 - \Lambda]^3}. \tag{28}$$

Taking $q_0 = 1$, $H_0 = 75$ km s^{-1} Mpc^{-1} and $\varepsilon_0 = 3.1\times10^{-31}$ g cm^{-3}, we find that

$$\frac{\sigma_0^2}{\Theta_0^2} = \frac{1867\times10^{-38}+\Lambda}{11250\times10^{-38}}. \tag{29}$$

For $\Lambda = 0$ we get $\sigma_0 \approx 0.4\Theta_0$. This is not in accord with observation (Ellis 1971). If we set the limit for σ_0 as $\sigma_0 < \frac{1}{4}\Theta_0$ we find that

$$-1.867\times10^{-38} < \Lambda < -1.164\times10^{-35}.$$

The universe in this case will obviously have expansion from an initial singularity for a finite period of time followed by contraction to another singularity.

We can completely integrate the equation (21) for $\gamma = 2$ and $\Lambda = 0$. The resulting metric can be transformed to

$$ds^2 = a\sqrt{\tau^2-1}\left\{dX^2 + e^{-2X}\left[\left(\frac{\tau-1}{\tau+1}\right)^{m/4a} dY^2 + \left(\frac{\tau-1}{\tau+1}\right)^{-m/4a} dZ^2\right]\right\} - \frac{a\, d\tau^2}{4\sqrt{\tau^2-1}} \tag{30}$$

with

$$8\pi G\varepsilon = 8\pi G p = \frac{8\pi G\varepsilon_0 R_0^6}{a^3} > 0 \tag{31}$$

where

$$a = \sqrt{\tfrac{1}{12}m^2 + \tfrac{8}{3}\pi G\varepsilon_0 R_0^6}.$$

The model has a singularity at $\tau = 1$. It is a point singularity if $m^2 < 16\pi G\varepsilon_0 R_0^6$. The singularity is of the cigar type or the barrel type according to whether $m^2 \gtrless 16\pi G\varepsilon_0 R_0^6$. The model starts expanding from its singular state and continues to expand till $\tau = \infty$, at which stage the space-time becomes flat.

The metric (15) is in general of Petrov type I. For type D we must have

$$\left(\frac{1}{R}\frac{dR}{dT}\right)^2 = R^4 + \tfrac{1}{4}m^2 \tag{32}$$

so that

$$R^2 = \tfrac{1}{2}m \operatorname{cosech}\{m(M-T)\} \tag{33}$$

where M is a constant. By suitable transformation of coordinates the metric (15) reduces to

$$ds^2 = \tfrac{1}{2}m\sqrt{\tau^2-1}\left\{dX^2 + e^{-2X}\left[\left(\frac{\tau-1}{\tau+1}\right)^{1/2} dY^2 + \left(\frac{\tau-1}{\tau+1}\right)^{-1/2} dZ^2\right]\right\} - \frac{m\, d\tau^2}{8\sqrt{\tau^2-1}}. \tag{34}$$

The pressure and density for (34) are given by

$$8\pi G p = 4/m(\tau^2-1)^{3/2} + \Lambda \tag{35}$$

$$8\pi G\varepsilon = 4/m(\tau^2-1)^{3/2} - \Lambda. \tag{36}$$

For $\Lambda = 0$ this is a special case of (30) with $a = \frac{1}{2}m$. The singularity in this case is of barrel type. The model (15) will be of Petrov type II if

$$R^2 = 1/2(b+T) \qquad m \neq 0 \tag{37}$$

b being a constant. However, the reality conditions $\varepsilon \geqslant p$, $p \geqslant 0$ are not satisfied.

We get another exact solution by assuming

$$\sigma/\Theta = K = \text{constant}. \tag{38}$$

The corresponding metric has the form

$$ds^2 = -d\tau^2 + \left(d + \frac{m}{2K}\tau\right)^{2/3}\left[dX^2 + e^{-2X}\left(d + \frac{m}{2K}\tau\right)^{2K} dY^2 + e^{-2X}\left(d + \frac{m}{2K}\tau\right)^{-2K} dZ^2\right] \tag{39}$$

by suitable transformations of coordinates, d being a constant. The pressure and density are given by

$$8\pi Gp = \frac{m^2(1-3K^2)}{12K^2(d+m\tau/2K)^2} + \frac{1}{(d+m\tau/2K)^{2/3}} + \Lambda \tag{40}$$

$$8\pi G\varepsilon = \frac{m^2(1-3K^2)}{12K^2(d+m\tau/2K)^2} - \frac{3}{(d+m\tau/2K)^{2/3}} - \Lambda. \tag{41}$$

Reality conditions $\varepsilon \geqslant p$ and $p \geqslant 0$ lead to

$$K^2 < \tfrac{1}{3} \qquad \Lambda < -4K\sqrt{3}/m\sqrt{1-3K^2}$$

$$\alpha_0 \leqslant (d+m\tau/2K)^{2/3} \leqslant -\tfrac{1}{2}\Lambda$$

α_0 being the positive root of the cubic equation

$$\frac{m^2(1-3K^2)}{12K^2}\alpha^3 + \alpha + \Lambda = 0. \tag{42}$$

The model is singular when $\tau = -2Kd/m$. By virtue of the reality condition this is a point singularity. The model satisfies the strong condition $\int_{t_0}^{\infty} \sigma_{\alpha\beta}\, dt$ = finite, so that it approximately becomes a Robertson–Walker metric (MacCallum 1971). Since the fluid flow is geodetic the geodesic deviation vector determines the relative flow pattern of the fluid. This vector η^i satisfies the equation

$$\frac{\delta^2\eta^i}{\delta s^2} + R^i_{jkl}V^j\eta^k V^l = 0. \tag{43}$$

For the metric (15) it reduces to the set

$$\begin{aligned} &\ddot{\eta}^\alpha + 2\dot{\eta}^\alpha(\tfrac{1}{3}\Theta + \sigma_{(\alpha)}) + \tfrac{2}{3}\sigma_{(\alpha)}\Theta = 0 \qquad \alpha = 1, 2, 3 \\ &\dot{\eta}^4 + \eta^4\Theta = \beta/R^3 \end{aligned} \tag{44}$$

where $\sigma_{(1)}, \sigma_{(2)}, \sigma_{(3)}$ are the eigenvalues of the shear tensor and β is a constant. Since $\sigma_{(1)}$ is zero we find that the effect of the shear is in the transverse components of η^i. On integration η^1 and η^4 are given by

$$\eta^1 = \int \frac{\alpha}{R^3}\, dt + \alpha \qquad \eta^4 = \frac{\beta t + b}{R^3}. \tag{45}$$

If we assume that $\eta^i V_i = 0$ then $\beta = b = 0$. For the case $\sigma/\Theta = K$, we have

$$\eta^1 = E(d+m\tau/2K)^{1/3} + F \tag{46}$$

$$\eta^2=\begin{cases}(d+m\tau/2K)^{1/6}[P(d+m\tau/2K)^{\sqrt{1-24K}/6}+Q(d+m\tau/2K)^{-\sqrt{1-24K}/6}] & K<\frac{1}{24}\\ (d+m\tau/2K)^{1/6}[P+Q\log(d+m\tau/2K)] & K=\frac{1}{24}\\ (d+m\tau/2K)^{1/6}P\sin\{\frac{1}{6}\sqrt{24K-1}\log(d+m\tau/2K)+\theta\} & K>\frac{1}{24}\end{cases} \tag{47}$$

$$\eta^3=(d+m\tau/2K)^{(6K+1)/(18K+6)}[L(d+m\tau/2K)^{[K(2K+1)/(3K+1)]^{1/2}} + M(d+m\tau/2K)^{-[K(2K+1)/(3K+1)]^{1/2}}] \tag{48}$$

where E, F, P, Q, L, M and θ are constants. Each component of η^i tends to a finite limit as one approaches the singularity. Moreover, the magnitude of the vector η^i also tends to zero.

3. Case II: $\mu_1=0$, $\nu_1\neq 0$

In this case equation (9) leads to $\nu_4=0$. From equation (10) we get

$$\frac{\mu}{A}\frac{\nu_1}{\nu}=f(t). \tag{49}$$

Equation (11) then leads to

$$2(A_4\mu-A\mu_4)_4=f(\nu_1/\nu). \tag{50}$$

The metric (1) in this case reduces to the form

$$ds^2=-dt^2+\phi^2\,dX^2+\mu(e^X\,dY^2+e^{-X}\,dZ^2) \tag{51}$$

after suitable transformation of coordinates, ϕ and μ being functions of t alone. The functions ϕ and μ are related by the equation

$$2\frac{\phi_{44}}{\phi}+\frac{\phi_4}{\phi}\frac{\mu_4}{\mu}-\frac{\mu_{44}}{\mu}=\frac{1}{\phi^2}. \tag{52}$$

This metric belongs to Bianchi type VI_0. The non-vanishing components of the Weyl tensor are

$$C_{1212}=e^X(-\tfrac{1}{12}\mu-\tfrac{1}{12}\mu(\phi^2)_{44}+\tfrac{1}{12}\phi^2\mu_{44}-\phi^2\mu_4^2/12\mu+\tfrac{1}{24}(\phi^2)_4\mu_4+\tfrac{1}{6}\mu\phi_4^2) \tag{53a}$$

$$C_{1313}=e^{-X}(-\tfrac{1}{12}\mu-\tfrac{1}{12}\mu(\phi^2)_{44}+\tfrac{1}{12}\phi^2\mu_{44}-\phi^2\mu_4^2/12\mu+\tfrac{1}{24}(\phi^2)_4\mu_4+\tfrac{1}{6}\mu\phi_4^2) \tag{53b}$$

$$C_{2323}=\tfrac{1}{6}(\mu^2/\phi^2+\mu_4^2-\mu\mu_{44}-\mu\mu_4\phi_4/\phi+2\mu^2\phi_{44}/\phi) \tag{53c}$$

$$C_{2124}=e^X(-\mu_4/\mu+\mu\phi_4/2\phi) \tag{53d}$$

$$C_{3134}=e^{-X}(\mu_4/\mu-\mu\phi_4/2\phi). \tag{53e}$$

The metric is in general of Petrov type I and does not have algebraically special subcases. However if $C_{1212}=C_{1313}=C_{2323}=0$ then the free gravitational field is of the magnetic type. In this case

$$2\mu\phi_{44}-\phi\mu_{44}+\phi(\mu_4^2/\mu)-\phi_4\mu_4+\mu/\phi=0. \tag{54}$$

Equations (52) and (54) lead to the following solutions

$$\begin{aligned}\phi &= \exp[-\tfrac{1}{2}(\phi_4^2-2)-(\tfrac{1}{2}\phi_4\sqrt{\phi_4^2-2}+\sinh^{-1}\sqrt{\tfrac{1}{2}\phi_4^2-1}+a)]\\ \mu &= \exp(b-\phi_4\sqrt{\phi_4^2-2}-\phi_4^2)\end{aligned} \qquad (a)$$

$$\begin{aligned}\phi &= \exp[-\tfrac{1}{2}(\phi_4^2-2)+(\tfrac{1}{2}\phi_4\sqrt{\phi_4^2-2}+\sinh^{-1}\sqrt{\tfrac{1}{2}\phi_4^2-1}+a)]\\ \mu &= k(\phi_4+\sqrt{\phi_4^2-2})^2\end{aligned} \qquad (b)$$

a, *b* and *k* being constants. After suitable transformations we get the metric corresponding to cases (*a*) and (*b*)

$$\begin{aligned}ds^2 &= \exp[-(T^2-2)-(T\sqrt{T^2-2}+2\sinh^{-1}\sqrt{\tfrac{1}{2}T^2-1})+b]\\ &\quad\times\left(dX^2-\frac{4dT^2}{((T^2-2)-T\sqrt{T^2-2})^2}\right)\\ &\quad+\exp(-T\sqrt{T^2-2}-T^2)(e^X\,dY^2+e^{-X}\,dZ^2)\end{aligned} \qquad (55)$$

and

$$\begin{aligned}ds^2 &= \exp[-(T^2-2)+(T\sqrt{T^2-2}+2\sinh^{-1}\sqrt{\tfrac{1}{2}T^2-1})+b]\\ &\quad\times\left(dX^2-\frac{4\,dT^2}{((T^2-2)+T\sqrt{T^2-2})^2}\right)\\ &\quad+(T+\sqrt{T^2-2})^2(e^X\,dY^2+e^{-X}\,dZ^2).\end{aligned} \qquad (56)$$

The pressure and density for the metric are given below. For metric (55)

$$8\pi Gp = -\tfrac{3}{2}\exp[(T^2-2)+(T\sqrt{T^2-2}+2\sinh^{-1}\sqrt{\tfrac{1}{2}T^2-1})-b](T^2+T\sqrt{T^2-2}-\tfrac{3}{2})+\Lambda \qquad (57)$$

$$8\pi G\varepsilon = \tfrac{3}{2}\exp[(T^2-2)+(T\sqrt{T^2-2}+2\sinh^{-1}\sqrt{\tfrac{1}{2}T^2-1})-b](T^2+T\sqrt{T^2-2}-\tfrac{1}{2})-\Lambda \qquad (58)$$

and for the metric (56)

$$8\pi Gp = -\tfrac{3}{2}\exp[(T^2-2)-(T\sqrt{T^2-2}+2\sinh^{-1}\sqrt{\tfrac{1}{2}T^2-1})-b](T^2-\tfrac{1}{3}T\sqrt{T^2-2}-\tfrac{3}{2})+\Lambda \qquad (59)$$

$$8\pi G\varepsilon = \tfrac{3}{2}\exp[(T^2-2)-(T\sqrt{T^2-2}+2\sinh^{-1}\sqrt{\tfrac{1}{2}T^2-1})-b](T^2+T\sqrt{T^2-2}-\tfrac{1}{2})-\Lambda. \qquad (60)$$

The reality conditions for the first model require that

$$\begin{aligned}&3\exp[(T^2-2)+(T\sqrt{T^2-2}+2\sinh^{-1}\sqrt{\tfrac{1}{2}T^2-1})-b](T^2+T\sqrt{T^2-2}-\tfrac{3}{2})<2\Lambda\\ &\quad<3\exp[(T^2-2)+(T\sqrt{T^2-2}+2\sinh^{-1}\sqrt{\tfrac{1}{2}T^2-1})-b]\\ &\quad\times(T^2+T\sqrt{T^2-2}-1).\end{aligned} \qquad (61)$$

Similarly for the second model we must have

$$\begin{aligned}&3\exp[(T^2-2)-(T\sqrt{T^2-2}+2\sinh^{-1}\sqrt{\tfrac{1}{2}T^2-1})-b](T^2+\tfrac{1}{3}T\sqrt{T^2-2}-1)>2\Lambda\\ &\quad>3\exp[(T^2-2)-(T\sqrt{T^2-2}+2\sinh^{-1}\sqrt{\tfrac{1}{2}T^2-1})-b]\\ &\quad\times(T^2-\tfrac{1}{3}T\sqrt{T^2-2}-\tfrac{3}{2}).\end{aligned} \qquad (62)$$

4. Conclusion

Starting with a general cylindrically symmetric metric for which the time lines are geodesics, we find that when the transverse scale factors at each point have a ratio independent of the spatial coordinate, the metric admits an additional Killing vector and the space–time becomes homogeneous. Similarly when they are inversely proportional, the factor of proportionality being a function of t alone, the space–time becomes homogeneous. In the case of a Bianchi type V universe the anisotropy is in the transverse directions and it becomes increasingly large as one approaches the singularity. The behaviour of the anisotropy is also clear from the nature of the geodesic deviation vector which shows that the anisotropy plays a role only in the transverse components while the axial component is independent of it. The magnitude of this vector tends to zero as one approaches the singularity.

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